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Letters to the Editor

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Abstract. We report a natural all-order decorrelation procedure for the interaction of a single two-level atom with all modes of the radiation field which replaces the semiclassical boson approximation for the atom by an exact fermion treatment. In consequence, the spontaneous emission of the semiclassical theory is extended to include stimulated emission, a semiclassical Lamb-type shift is explicitly replaced by a shift satisfying the Bethe formula manifestly due to vacuum fluctuations, and there can be two intensity-dependent terms, one of which generalizes the Bethe formula.

From the Hamiltonian density for the dipole interaction

$$H_{\text{int}}(\boldsymbol{x}, t) = -e\boldsymbol{r}(\boldsymbol{x}, t).\boldsymbol{e}(\boldsymbol{x}, t) \tag{1}$$

in which $\boldsymbol{r}(\boldsymbol{x}, t)$ is the dipole density operator and $\boldsymbol{e}(\boldsymbol{x}, t)$ is a field operator (both operators taken in Heisenberg representation), it is possible to reach the classical integral equation for the classical dipole density

$$\boldsymbol{P}(\boldsymbol{x}, \omega) \equiv \sum_i \delta(\boldsymbol{x}-\boldsymbol{x}_i)\boldsymbol{P}_i(\omega) \qquad \text{at frequency } \omega$$

$$\boldsymbol{P}(\boldsymbol{x}, \omega) = \boldsymbol{\alpha}(\omega).\sum_i \delta(\boldsymbol{x}-\boldsymbol{x}_i)\{\boldsymbol{E}_{\text{ext}}(\boldsymbol{x}, \omega) + \int \mathbf{F}(\boldsymbol{x}, \boldsymbol{x}'; \omega).\boldsymbol{P}(\boldsymbol{x}', \omega)\,\mathrm{d}\boldsymbol{x}'\} \tag{2}$$

(cf. Bullough *et al.* 1968, Obada and Bullough 1969). Here $\boldsymbol{\alpha}(\omega)$ is the Kramers–Heisenberg polarizability tensor and $\boldsymbol{P}_i(\omega)$ is the dipole (at frequency ω) induced in the ith particle located at $\boldsymbol{x}_i$ by the field $\boldsymbol{E}_{\text{ext}}(\boldsymbol{x}, \omega)$. This field is 'external': it is a semiclassical field used to probe the system of matter and second quantized field coupled by the dipole interaction (1). Consequently, although (2) depends on all orders of the quantized field $\boldsymbol{e}(\boldsymbol{x}, t)$, it is linear in the probe $\boldsymbol{E}_{\text{ext}}(\boldsymbol{x}, \omega)$.

The photon propagator $\mathbf{F}(\boldsymbol{x}, \boldsymbol{x}'; \omega)$ is the classical quantity

$$\begin{aligned} \mathbf{F}(\boldsymbol{x}, \boldsymbol{x}'; \omega) &\equiv (\nabla\nabla + k_0^2\mathbf{U})\exp(\mathrm{i}\,k_0|\boldsymbol{x}-\boldsymbol{x}'|)|\boldsymbol{x}-\boldsymbol{x}'|^{-1} \\ k_0 &\equiv \omega c^{-1}. \end{aligned} \tag{3}$$

$\mathbf{U}$ is the unit tensor: it describes the Hertz dipole field at $\boldsymbol{x}$ due to the source $\boldsymbol{P}(\boldsymbol{x}', \omega)\,\mathrm{d}\boldsymbol{x}'$ at $\boldsymbol{x}'$. It emerges from the field $\boldsymbol{e}(\boldsymbol{x}, t)$ in (1) (developed in interaction representation) because of the result of Jordan and Pauli (1928) that the unequal space–time commutator of two *free*-field operators is a c number. The causal part of the Fourier transform of this is precisely $\mathbf{F}(\boldsymbol{x}, \boldsymbol{x}'; \omega)$:

$$\int_{-\infty}^{\infty} \mathrm{i}\,\hbar^{-1}[\boldsymbol{e}(\boldsymbol{x}, t), \boldsymbol{e}(\boldsymbol{x}', t')]\theta(t-t')\exp\{-\mathrm{i}\,\omega(t-t')\}\,\mathrm{d}t' = \mathbf{F}(\boldsymbol{x}, \boldsymbol{x}'; \omega) \tag{4}$$

($\theta(\tau)$ is the step function).

A crucial feature of the argument which reduces the interaction (1) to the solution of (2) is a systematic decorrelation procedure which treats the dipole operators $e\boldsymbol{r}_i(t)$ as boson operators. We specialize the discussion to spinless two-level atoms with non-degenerate states $|0\rangle$ (the ground state) and $|s\rangle$ and energies E_0 and E_s: $E_s - E_0 \equiv \hbar\omega_s$. We choose the states so that $\boldsymbol{r}_{0s} = \boldsymbol{r}_{s0} = x_{0s}\boldsymbol{u}$ (say) where $\boldsymbol{u}$ is a unit vector. At $t = 0$ we can then set

$$e\boldsymbol{r} = ex_{0s}\boldsymbol{u}(\sigma_+ + \sigma_-) \tag{5}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y)$ and the matrices σ_x, σ_y are Pauli spin matrices. The annihilation and creation operators σ_- and σ_+ satisfy the fermion commutation and anti-commutation relations $[\sigma_-, \sigma_+] = \sigma_z$, $[\sigma_-, \sigma_+]_+ = 1$. As boson operators they would need to satisfy $[\sigma_-, \sigma_+] = 1$. In this case, with $\sigma_x = \sigma_+ + \sigma_-$ in interaction representation with respect to the free particle Hamiltonian $H_0 \equiv \frac{1}{2}\hbar\omega_s\sigma_z$, the commutator $\mathrm{i}\hbar^{-1}[\sigma_x(t), \sigma_x(t')]$ is also a c number and there is a polarizability which proves to be

$$\int_{-\infty}^{\infty} e^2x_{0s}^2\,\mathrm{i}\hbar^{-1}[\sigma_x(t), \sigma_x(t')]\theta(t-t')\exp\{-\mathrm{i}\,\omega(t-t')\}\,\mathrm{d}t'$$
$$= e^2x_{0s}^2\hbar^{-1}2\omega_s(\omega_s^2-\omega^2)^{-1} \tag{6}$$

exactly the Kramers–Heisenberg result. The integral equation (2) now follows *exactly* for a system of many two-level atoms satisfying boson commutation relations; although these are incorrect the classical result (2) is held to be a good approximation for few photons inducing few excitations amongst many atoms (cf. e.g. Hopfield 1958): it is presumably good, too, for an almost fully inverted dielectric with a very few atoms in their ground states and very few photons.

For a single particle at $\boldsymbol{x}_i$, (2) is immediately but only formally soluble: it reduces to

$$\boldsymbol{P}_i\delta(\boldsymbol{x}-\boldsymbol{x}_i) = \alpha(\omega)\,\delta(\boldsymbol{x}-\boldsymbol{x}_i)(\boldsymbol{E}_{\mathrm{ext}}(\boldsymbol{x}, \omega) + \mathbf{J}_0 . \boldsymbol{P}_i) \tag{7a}$$

when $\boldsymbol{E}_{\mathrm{ext}}(\boldsymbol{x}, \omega)$ is parallel to the unit vector $\boldsymbol{u}$ of (5). Here

$$\mathbf{J}_0(\omega) \equiv \int \mathbf{F}(\boldsymbol{x}, \boldsymbol{x}'; \omega)\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' = J_0(\omega)\mathbf{U} \tag{7b}$$

and must be a scalar multiple of the unit tensor if it exists at all; thus the solution of (7a) is then

$$\boldsymbol{P}_i\delta(\boldsymbol{x}-\boldsymbol{x}_i) = \gamma(\omega)\,\delta(\boldsymbol{x}-\boldsymbol{x}_i)\boldsymbol{E}_{\mathrm{ext}}(\boldsymbol{x}, \omega)$$
$$\gamma(\omega) = \alpha(\omega)(1-J_0(\omega)\,\alpha(\omega))^{-1}. \tag{8}$$

The scalar quantity $J_0(\omega)$ is divergent with convergent part precisely $\frac{2}{3}\mathrm{i}k_0^3$. The interpretation $J_0(\omega) \equiv \frac{2}{3}\mathrm{i}k_0^3$ is essential for internal consistency in the optical scattering theory (Bullough *et al.* 1968, Bullough and Hynne 1968). Further, for the two-level atom, $\gamma(\omega)$ has a resonant scattering width which is the Weisskopf–Wigner (1930) spontaneous emission width $\Gamma_0 = \frac{4}{3}e^2x_{0s}^2\hbar^{-1}\omega_s^3c^{-3}$ since

$$\gamma(\omega) = e^2x_{0s}^2\hbar^{-1}2\omega_s(\omega_s^2-\omega^2-\tfrac{4}{3}\mathrm{i}k_0^3e^2x_{0s}^2\omega_s\hbar^{-1})^{-1} \tag{9a}$$
$$\sim e^2x_{0s}^2\hbar^{-1}(\omega_s-\omega-\tfrac{2}{3}\mathrm{i}\omega_s^3c^{-3}e^2x_{0s}^2\hbar^{-1})^{-1} \tag{9b}$$

close to resonance. Thus the spontaneous emission is a classical result ((8) with (9) is obviously the driven classical Lorentz oscillator) and the Lamb shift is either concealed in the divergent part of $J_0(\omega)$ or has been wholly excluded from the theory by the boson assumption for the dipole operators. In fact the $J_0(\omega)$ given by (7*b*) is of this form and infinite apparently only because of the assumption of a dipole interaction in (1). A dipole approximation cannot be acceptable for a one-particle self-energy, but when $J_0(\omega)\,\alpha(\omega)$ is corrected the theory coincides with the semiclassical Lamb shift theory recently reported by Crisp and Jaynes (1969).

Although (2) follows exactly from a second quantized theory taken to all orders in the internal field operators $e(x, t)$ (once the boson assumption for the dipoles is made) it is plain that there can be no stimulated emission: for only the c number commutator of two field operators with the transform (4) appears in the theory and this cannot depend on the field states. But it is also clear that the classical expression (2) cannot be a good approximation to any second quantized one-atom theory when it is valid only for a few excitations compared with the number of particles. We have therefore obtained a natural analogue of (7) for a single non-degenerate two-level atom which retains the fermion commutation relations for the dipole operators.

It is not sufficient now simply to appeal to the boson character of the photons and, for a tractable calculation, products of field operators must be decorrelated. We find that there is a natural choice for doing this which, however, exhibits the field operators as the expectation value of the anti-commutator of the fields

$$\mathrm{i}\hbar^{-1}\langle \mathrm{ph}|[e(x,t), e(x',t')]_+|\mathrm{ph}\rangle. \qquad (10)$$

This depends on the field states $|\mathrm{ph}\rangle$. At the same time, beyond first order, $\alpha(\omega)$ becomes the Fourier transform of a c number anti-commutator:

$$\alpha(\omega) \to \alpha_+(\omega) = 2\omega e^2 x_{0\mathrm{s}}{}^2 \hbar^{-1}(\omega_\mathrm{s}{}^2-\omega^2)^{-1} \qquad (11)$$

(note the ω in the numerator compared with (6)). Next, $\gamma(\omega)$ changes to

$$\gamma_+(\omega) = -2\omega_\mathrm{s} e^2 x_{0\mathrm{s}}{}^2\hbar^{-1}(\rho_{\mathrm{ss}}-\rho_{00})(\omega_\mathrm{s}{}^2-\omega^2-2\omega e^2 x_{0\mathrm{s}}{}^2\hbar^{-1}J_0{}^+(\omega))^{-1} \qquad (12)$$

where $\rho_{\mathrm{ss}}-\rho_{00} = -1$ for atoms initially in their ground states and is $+1$ for those initially in their upper states.

The scalar number $J_0{}^+(\omega)$ depends on the frequency $\omega = ck_0$ of the external field and on the field occupation numbers n_k of the total system (matter plus field) probed by that external field. We find that

$$4\omega e^2 x_{0\mathrm{s}}{}^2\hbar^{-1}\mathrm{Im}(J_0{}^+(\omega)) = (n_{k_0}+1)\tfrac{4}{3}e^2\hbar^{-1}k_0{}^3 x_{0\mathrm{s}}{}^2 \qquad (13)$$

where n_{k_0} is the (isotropic) field occupation number at $|k| = k_0$. Close to resonance this means that (9) has the resonance width $(n_{k_0}+1)\Gamma_0$ (if we can simply set $\omega \to \omega_\mathrm{s}$ in (13)). Thus the stimulated emission is now appearing in conventional form.

The real part of $J_0{}^+(\omega)$ depends on all the n_k; but when, for example, $n_k = |\alpha|^2 =$ constant $a < k < b$, $= 0$ otherwise (and is isotropic with $b > \omega$)

$$2\omega e^2 x_{0\mathrm{s}}{}^2\hbar^{-1}\,\mathrm{Re}(J_0{}^+(\omega)) = -e^2 x_{0\mathrm{s}}{}^2\hbar^{-1}2\omega(4/3\pi)k_0{}^3\ln(m_\mathrm{e}c^2/\hbar\omega)$$
$$-e^2 x_{0\mathrm{s}}{}^2\hbar^{-1}2\omega(2/3\pi)[k_0|\alpha|^2(b^2-a^2)+k_0{}^3|\alpha|^2\ln\{(c^2b^2-\omega^2)/|\omega^2-c^2a^2|\}]. \qquad (14)$$

In order to reach the first logarithmic term we have cut off the integration at the reciprocal Compton wavelength $k_\mathrm{C} = m_\mathrm{e}c\hbar^{-1}$. If it is possible simply to set $\omega \to \omega_\mathrm{s}$ in (14), the correction to $\hbar\omega_\mathrm{s}$ is

$$\Delta E^{(1)} \simeq -e^2 x_{0\mathrm{s}}{}^2(4/3\pi)\omega_\mathrm{s}{}^3c^{-3}\ln(m_\mathrm{e}c^2/\hbar\omega_\mathrm{s})$$
$$-e^2 x_{0\mathrm{s}}{}^2(2/3\pi)[\omega_\mathrm{s}c^{-1}|\alpha|^2(b^2-a^2)+\omega_\mathrm{s}{}^3c^{-3}|\alpha|^2\ln\{(c^2b^2-\omega_\mathrm{s}{}^2)/(\omega_\mathrm{s}{}^2-c^2a^2)\}] \qquad (15)$$

(for $cb > \omega_s > ca$). The first logarithmic term is twice that appearing in the non-relativistic Bethe (1947) formula for the Lamb shift. This is to be expected since Bethe's shift $\Delta E_B^{(1)}$ corrects both E_s and E_0: since both levels are non-degenerate $E_s \to E_s - \Delta E_B^{(1)}$, $E_0 \to E_0 + \Delta E_B^{(1)}$. The second logarithmic term generalizes the Lamb shift to include the effect of field occupation numbers. Evidently it vanishes for a symmetric distribution of field modes for which $c^2b^2 - \omega_s^2 = \omega_s^2 - c^2a^2$. The remaining term does not so vanish: it obviously generalizes the electromagnetic mass shift in the vacuum to include the dependence on photon occupation numbers, but (see below (18)) should be observable in a sufficiently intense *isotropic* field.

Three other results are of interest: first since the Bethe form of the Lamb shift appears in (12) in consequence of (15) (whether real photons are present or not) we do not now expect any semiclassical contribution to this shift of the type occurring in (9). In fact that (linearly) divergent part of (7*a*) apparently identifiable with the term of Crisp and Jaynes (1969) is explicitly eliminated in (12). It is replaced by two terms both manifestly depending on vacuum fluctuations since

$$[e(\boldsymbol{x}, t), e(\boldsymbol{x}', t')]_+ = [e(\boldsymbol{x}, t), e(\boldsymbol{x}', t')] + 2e(\boldsymbol{x}', t')\, e(\boldsymbol{x}, t). \tag{16}$$

One term is the logarithmic Bethe term, the other is quadratically divergent. Such divergence is associated with the A^2 term (effectively concealed in the interaction (1)) in a theory of the Lamb shift of the ground state reported elsewhere (Bullough 1969). Both the linearly divergent (semiclassical) and quadratically divergent terms there accompany the logarithmic Bethe term and are interpreted as mass renormalizations.

A second result is a classical limit of the theory. We consider a single mode at the atomic resonance frequency ω_s with the same polarization $\boldsymbol{u}$ as the external field: the classical amplitude is $E(\omega)$ so that

$$n_k = \frac{2\pi^2 E^2(\omega_s)}{\hbar\omega_s}\,\delta(\boldsymbol{k} - k_s\hat{\boldsymbol{k}}_0). \tag{17a}$$

Ignoring both the damping and the vacuum terms we find that

$$\gamma_+(\omega) = \frac{e^2 2\omega_s \hbar^{-1} x_{0s}^2(\omega_s^2 - \omega^2)}{(\omega_s^2 - \omega^2)^2 - 4\omega^2 e^2 x_{0s}^2 E^2(\omega_s)\hbar^{-2}}. \tag{17b}$$

This resonates at $\omega \simeq \omega_s \pm w_1$ with an effective Rabi frequency $w_1 = ex_{0s}\hbar^{-1} E(\omega_s)$. This result agrees with that we can obtain for a single two-level atom coupled to a strong semiclassical field (this problem is soluble in closed form within the 'rotating-wave approximation' in which $\omega_s^2 - \omega^2 \simeq 2\omega_s(\omega_s - \omega)$ (e.g. Lamb 1964).

More generally, if the classical modes are distributed with the number $c^{-1}g(\omega' - \omega_s)\,d\omega'$ in $d\omega'$ we find that, in the rotating-wave approximation, ω_s shifts to $\omega_s - \hbar^{-1}\Delta E$ where (P denotes principal value)

$$\hbar^{-1}\Delta E = e^2\hbar^{-2}x_{0s}^2\{\mathrm{P}\int E^2(\omega')g(\omega' - \omega_s)(\omega' - \omega)^{-1}\,d\omega' + i\pi E^2(\omega)\,g(\omega - \omega_s)\} \tag{18}$$

—results which are comparable with those of Brossel (1964). The real energy shift vanishes for $\omega = \omega_s$ and symmetric $g(\omega' - \omega_s)\,E^2(\omega')$ and there is no term like the quadratic term in (15). This is a consequence of the extremely anisotropic character of field occupations like (17*a*).

Thirdly, we note that if $\gamma_+(\omega)$ is in any sense an effective one-particle polarizability in a many-particle system then both (13) and (18) imply exponential growth, with an exponent proportional to the intensity, in an amplifying medium (for which $\rho_{ss}-\rho_{00} = 1$).

The theory will be published in detail elsewhere.

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Comments on the paper *High transverse momenta observed in air shower cores*

Abstract. The recent findings of the Sydney air shower group concerning the existence of high-transverse-momentum events in air shower cores are disputed. It is shown that the multiple cores which are interpreted as large-transverse-momentum events are likely to be simulated by the detection fluctuations quoted by the Sydney group itself.

The Sydney air shower group (McCusker *et al.* 1969, Bakich *et al.* 1969, Bakich *et al.* 1970 to be referred to as I, II, and III respectively) have claimed that they have found evidence for the existence of large-transverse-momentum events in air shower cores. The transverse momenta (p_t) reported range up to more than 100 GeV/c and it was these findings which lead them to their widely known quark search. The Sydney evidence for large p_t is based primarily on the observation of multiple cores in the electron distribution of showers recorded by means of a 16 m^2 scintillation counter hodoscope consisting of 64 quadratic elements. Shower cores with well separated peaks have been observed and, using electromagnetic cascade theory, the transverse momenta necessary to explain the observed separation of peaks have been estimated.

Since 1965 we have recorded shower cores in the same region of primary energy (10^{14} to 10^{17} eV) employing a 32 m^2 neon hodoscope. This detector has an excellent stability and uniformity of response and we are able to measure particle densities up